I have been revising our course Foundations of Mathematics, which is the gateway to the major. Sections are 20 to 30 students. The courses' goals are to increase students' abilities to construct proofs, solve nonstandard problems, and communicate these solutions and proofs both orally and in written form.

I have found this course extremely challenging to teach because I, like most mathematicians, instinctively know what is a likely approach to proving a result and how that proof should be presented. So I had difficulty understanding where students were having problems and how to help them develop appropriate problem solving strategies and writing skills. This has forced me to spend a good deal of time reflecting on my own thinking and writing.

In this talk, I want to share some of the techniques I have begun using. Some of these techniques resulted from my use of a student assessment tool, Think-alouds, in which students are videotaped while solving problems. I will talk more about this later.

I now construct lessons in blocks of time instead of topic of the day. A new topic is presented and the students are given a collection of related individual and group problems to work over the next several weeks. Each group has at least one problem that is different from the rest of the class.

Typical problems revisit calculus and limits as well as surveying other areas of math.

1) Consider the function

$$f(x) = \begin{matrix} x^2 & x \text{ rational} \\ 0 & x \text{ irrational} \end{matrix}$$

Show $f'(x)=0$.

2) Let G be a graph such that every cycle is of odd length. Let *e* be an edge in G. Claim: *e* appears in at most 1 cycle. a) What is given in usable form? b) What must be shown, in usable form? c) What is the obvious method of proof and why? d) Give that proof.

Each group is required to email me an early draft of the group problem usually at the end of the first week. Similarly, each student must email me a draft of their individual problem after about 10 days. Most students use Word with Equation Editor. I email the drafts back with comments and hints in red. I am somewhat picky about students' use of language, particularly near the beginning of the course. Students are penalized for not submitting drafts on time.

For the group problem, I require that one student type the original draft, a second student revise the first draft and the third group member types the final draft. When the final version is submitted, they must also submit the 2 earlier drafts. Each student in the group is responsible for typing one draft and must sign their draft. This gives additional impetous for students to critically revise another student's work, since they know I will be checking to see if revisions were made. If a student didn't make many revisions to the previous draft and there are mistakes in the final draft, they will be marked down.

At the end of the unit, each group must present their problem and solution to the class. I find that this is more successful when each group is presenting a problem that is new to the rest of the class, since they must insure the rest of the class understands the solution. I have found that the students seem to enjoy doing these presentations and are quite creative in how they present their solutions. Seeing others' presentations help students learn different ways of approaching problems. When students organize the material to make a clear presentation, they reflect more on their own understanding.

I have one group evaluate the style of another's presentation and a second group to evaluate the content of the proof, looking for minor and major errors.

Although it is time consuming, I sometimes have groups give a dry run of the presentation to me during office hours. This has helped prevent embarrassing errors and the students often gain understanding.

I still have difficulty writing questions that are at an appropriate level and strike the right balance between giving students help versus giving away the answer.

I have been refining these techniques through the use of Think-alouds. Each time I teach this course, I get 3 student-volunteers from my class. Each of the students is videotaped for about 15 minutes working a moderately difficult problem on their own. While being videotaped, they are encouraged to talk about what they are thinking, how they are approaching the problem, and why they are using an approach. They are given hints when stuck. Then the students are brought together to continue working on the problem as a group. This is done with the same students at the beginning of the semester, near the middle of the semester, and at the end of the semester, so as to document progress.

The Think-alouds have helped me better understand how students think, why they get stuck, and what helps them change to a more productive approach.

Several observations that this organization addresses.

First: Students get stuck at a much earlier point in the problem then I expected. They seem to be afraid to take a first step if they are unsure what the second step is going to be. Class time is often spent with groups working on problems, so that they learn how to begin problems. TA's are also available in the evenings.

Second: Students have difficulty understanding what it is that they are supposed to do. They confuse what is given with what is to be shown. I have structured my problems so that students learn to write what is given in a usable form and what is to be shown in a usable form, such as

Problem: Show that if n is an odd integer, then n^2 is odd.

Given: There exists an integer k such that $n=2k+1$.
Show: There exists an integer j such that $n^2=2j+1$

This requires that they use definitions instead of intuition.

Third: This was surprising. A lack of progress is not necessarily a result of lack of effort. The videotaped students would work for long periods of time, even if they were not making progress.

Related to this is
Fourth: Student's have difficulty changing directions, even when it is clear that their current approach isn't working. They sometimes remind me of a windup toy car that is stuck in a corner, and can't turn around.

This is why I have instituted a draft system so that students don't waste a lot of time unproductively.

Fifth: When students are working together, they can solve problems that none of them could solve independently. This is why I use group work extensively.

Sixth: Students have difficulty understanding and using mathematical terminology, particularly "there exists", "for some" and "for every". At the beginning of the course I focus on just interpretation of problems. For example, I might give them a situation and several statements. They are to determine the truth of each statement in that situation. A simple example is the following

Determine which of statements p and q are true for each equation.
p= {For every x, there exists a y such that the equation is satisfied}
q= {There exists a y such that for every x, the equation is satisfied}
a) $2x+y=-2$
b) $x+y^2=3$
c) $(x-2)(y+3)=0$

I am finding that my students are developing their analytic abilities much more than they did when I began teaching this course. Part of this is because I have a better understanding of where their difficulties arise from watching them work problems, and have developed a supportive system in which they can constantly get their work evaluated and re-evaluated.

Most students have had little experience doing proofs beyond an intuitive, hand-waving level. I have therefore had to develop more realistic expectations about what can be accomplished in one semester. To be truly successful, we have to build on what they have learned in later courses.