The angle sum theorem for triangles on the sphere

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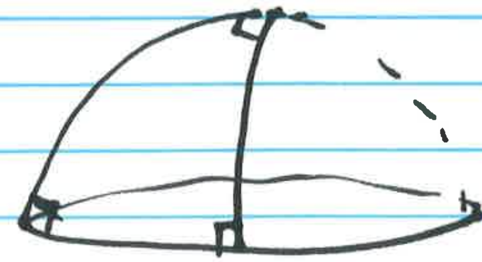
Theorem In a spherical triangle, the sum of the measures of the angles is greater than 180° (π radians)



Proof(s) (Traditional)

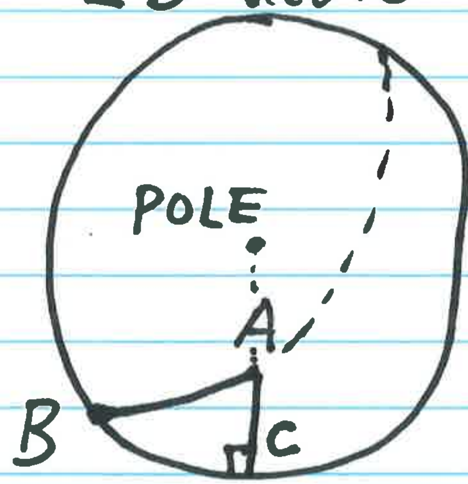
- Solid geometry

- Areas

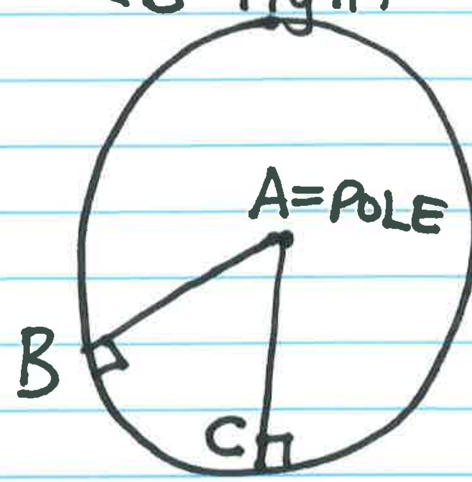
<u>Proposition</u>. If spherical $\triangle ABC$ has a right angle at C, then $\angle B$ and opposite side $\widehat{AC}$ are either both acute, both right or both obtuse. (Same for $\angle A$.)

<u>Pf</u>. Three cases: Top view of sphere.

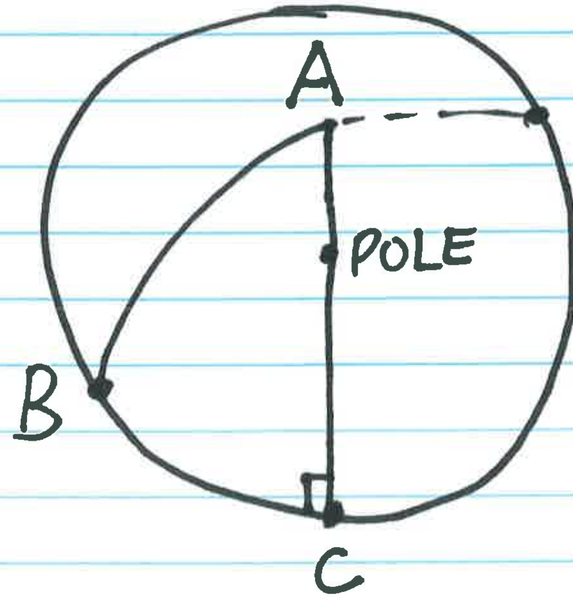
$\angle B$ acute



$\angle B$ right



$\angle B$ obtuse



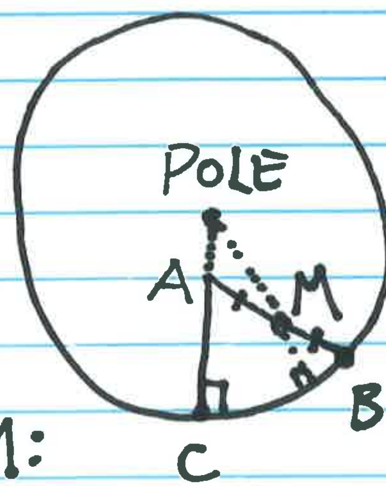
__Proposition.__ In a spherical __right__ triangle, the sum of the measures of the angles $> 180°$.

__PF__

WLOG $\angle A, \angle B$ acute.

So $\overset{\frown}{AC}, \overset{\frown}{BC}$ acute.

P = pole of $\overset{\frown}{BC}$



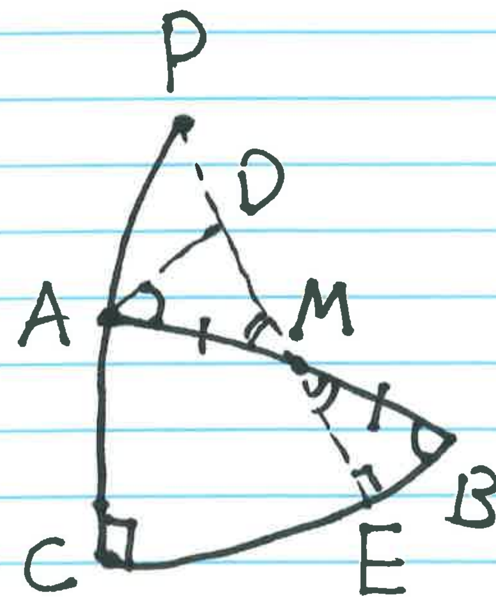
M = mid point $\overset{\frown}{AB}$

Extend $\overset{\frown}{PM}$ to $\odot CB$.

Build $\angle B$ inside $\angle PAM$:

$\triangle AMD \cong \triangle BME$ (ASA)

$\Rightarrow \angle ADP$ right



$\overset{\frown}{PD}$ acute

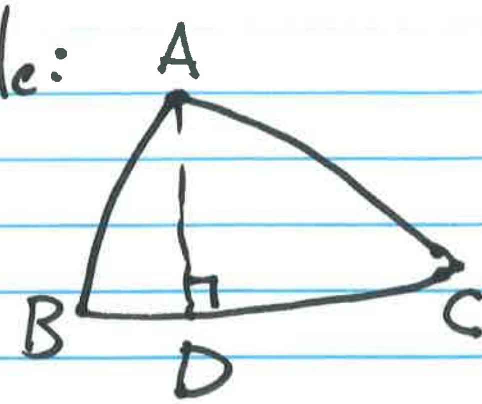
$\Rightarrow \angle PAD$ acute

$\Rightarrow \angle CAD$ obtuse

$\Rightarrow m\angle A + m\angle B > 90$

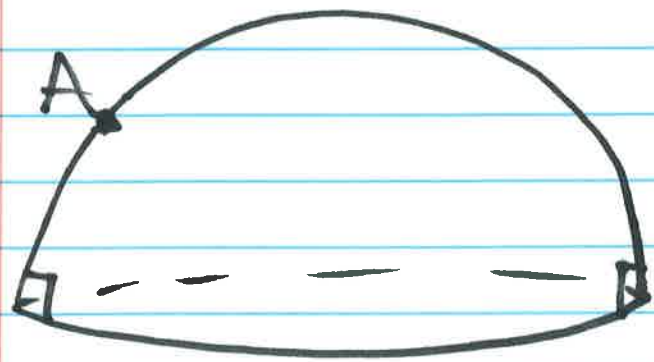
$\Rightarrow m\angle A + m\angle B + m\angle C > 180$

For general spherical triangle: drop altitude.

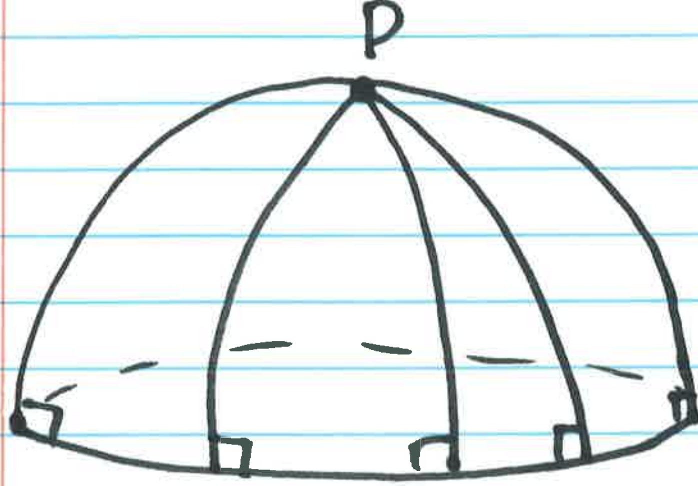


But how do we know such an altitude exists?

Perpendiculars on sphere:



Two perpendiculars from A to great circle

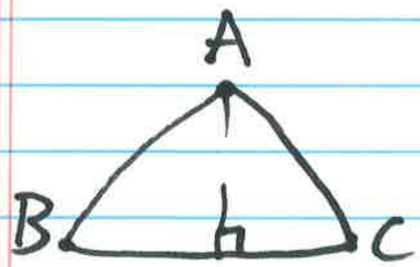


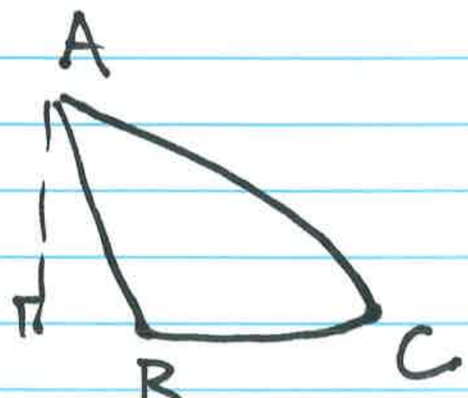
Infinitely many perpendiculars

Proposition

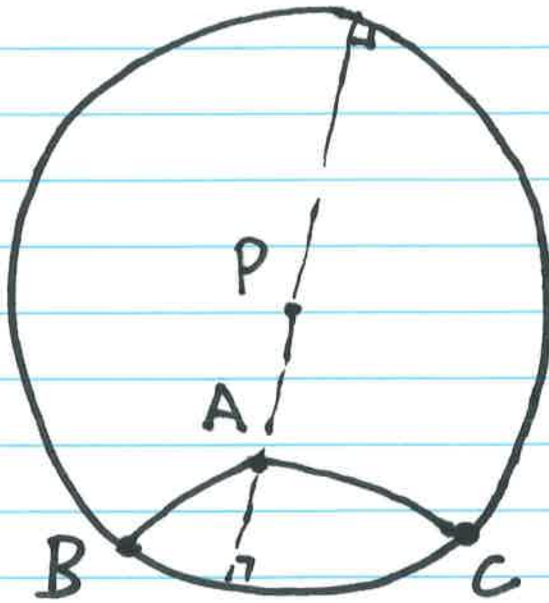
An altitude from a vertex of an angle of a spherical triangle exists which meets the interior of the opposite side

$\Leftrightarrow$ the other two angles are both acute, both right or both obtuse

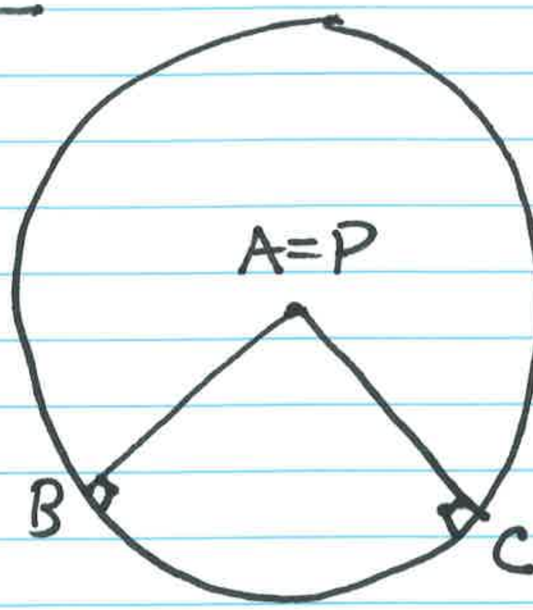




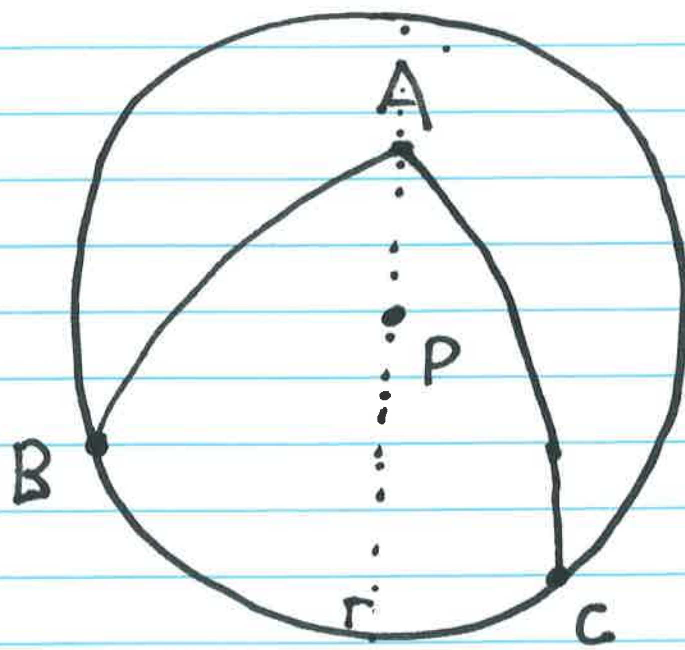
TOP VIEW



$\angle B$, $\angle C$ acute



$\angle B$, $\angle C$ right



$\angle B$, $\angle C$ obtuse