MAA Session
Bridging the Gap: Designing an Introduction to Proofs Course
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The Evolution of an Introduction to Proofs Course

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Original Assumptions

1. Having an entire 10-week course that carefully develops the basics about set properties, functions, and relations will solve the problem of student unpreparedness for courses like abstract algebra and real analysis.

2. Including a little discussion of logical principles is a good idea, but it is not something with which students will have difficulty.

3. If a mathematical fact seems trivial to me, it will seem trivial to my students. They understand the reasons why certain very basic mathematical facts are true.

4. If I speak clearly, correctly, and not too fast, students will understand what I say.

Students' Previous Experience

1. How to Get Right Answers without Understanding

   FOIL   Solve: $3 - 4x > 0$ Solve $(x - 3)(x + 1) > 0$

   Vertical and horizontal line tests

   $$\frac{d}{dx}\left(\int_0^x \sqrt{\cos(t^2)}\,dt\right) = ?$$ Find $f^{-1}(x)$ if $f(x) = \dfrac{1}{x^3 - 1}$.

2. Geometry without proof
3. Language: Everyday interpretations vs. logical interpretations
   a. if-then *vs.* if and only if *vs.* and
   b. there is… for every

Response to an intro to proofs course: Shock!

Prepare for Surprises – A Sampling

1. Is the product of two odd integers always odd, always even, or sometimes odd and sometimes even? How do you know for sure?

2. Use a truth table to show that a particular argument is invalid.

p	*q*	*r*	$\sim p$	$\sim p \vee q$	$p \to r$	$q \to p$	*r*
T	T	T	F	T	T	T	T
T	T	F	F	T	F	T	
T	F	T	F	F	T	T	
T	F	F	F	F	F	T	
F	T	T	T	T	T	F	
F	T	F	T	T	T	F	
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	F

Explanation: The 8th row of this truth table shows that it is possible for an argument of this form to have all true premises and a false conclusion. So this form of argument is invalid.

Student complaint: But you told us that a valid argument has true premises and a true conclusion.

Prepare for Surprises – A Sampling

1. Is the product of two odd integers always odd, always even, or sometimes odd and sometimes even? How do you know for sure?

2. Use a truth table to show that a particular argument is invalid.

3. If R is a relation on a set A, what does it mean for R to be symmetric?

4. Read the following out loud: $\{x \in \mathbf{R} \mid x^2 > 4\}$.

5. Is $\sqrt{3}$ irrational? How do you know?

6. Consider the sentence "The negative of any rational number is rational." Write it as $\forall$ x, if x _______ then, x _______.

7. Prove that the sum of any two odd integers is even.

Moral: Listen to your students!

Responses to Surprises – A Sampling

1. Practice applying logical principles. (Ex: taking negations of if-then and quantified statements and applying De Morgan's laws) and have exercises to rewrite quantified statements.

2. Reference logical principles as they arise later in the course. (Ex: two forms of the definition of one-to-one function)

3. Definitions: always have students explore both examples and non-examples. Emphasize the use of a definition as a test. (Ex: Is 0 even?) Focus on the role of definition in proofs.

4. Occasionally include parenthetical comments in proofs to explain the underlying logic.

5. Emphasize the role of variables as placeholders. (Ex: express definitions both with and without the use of variables; point out use of existential instantiation)

Direct Proof: What To Ask Yourself

1. Where do I start? What am I assuming?
2. What must I do to finish the proof?
3. How do I show that?

Change answers to 1-3 for contradiction and contraposition proofs.

Comment: Start with proofs where question 3 is relatively easy to answer. Recognize that as proofs get more and more advanced, answering question 3 becomes more and more challenging.

a. Prove that the sum of any two even integers is even.

b. Prove the transitivity of divisibility.

c. Prove that a composition of onto functions is onto.

d. Prove that $\sup_{x>0}\left(\frac{x}{x+1}\right)=1$.

e. Let A and B be bounded sets of real numbers and define and C = {a + b | a ∈ A and b ∈ B}. Prove that

$$\sup(C) = \sup(A) + \sup(B).$$

A Recent Experience

Prove by contradiction. For all integers n, if n^2 is even then n is even.

Student work. $(2k+1)^2 = 2k$

One of my comments. You can't have k mean two different things in the same equation.

Afterthought.

$$\frac{d(x^2)}{dx} = 2x$$

Comment. None of the x's in this equation is a variable!

How much attention do we pay to teaching students the real meaning of the "mathematical slang" we employ? How much do we confuse them by not doing so?

What we say	What we mean
the value of x	the quantity that is put in place of x
as the value of x increases	as larger and larger numbers are put in place of x
As the value of x increases, the value of y increases.	If larger and larger numbers are put in place of x, the corresponding numbers that are put in place of y become larger and larger.
where x is any real number	for all possible substitutions of real numbers in place of x
Let n be any even integer.	Imagine substituting an integer in place of n but do not assume anything about its value except that it is an even integer.
By definition of even, n = 2k for some integer k.	By definition of even, there is an integer we can substitute in place of k so that the equation n = 2k will be true. (Note that there is only one such integer; its value is n/2.)
the function x^2	the function that relates each real number to the square of that number. In other words, for each possible substitution of a real number in place of x, the function corresponds the square of that number.
where x is some real number that satisfies the given property	There is a real number that will make the given property true if we substitute it in place of x.
A general linear function is a function of the form f(x) = ax + b where a is any real number and b is any real number.	A general linear function is a function defined as follows: for all substitutions of real numbers in place of a and b, the function relates each real number to a times that number plus b. Or: the function is the set of ordered pairs where any real number can be substituted in place of the first element of the pair and the second element of the pair is a times the first number plus b.

Bridging the Gap: Both Ends Are Important

1. We know that if $f'(x)$ is positive, then the graph of f is increasing at x. Some people think that the converse of this statement is true. In other words, they believe that if the graph of f is increasing at x, then $f'(x)$ is positive.

a. Is this true or false? Explain. **b.** Show that this is false.

c. Give an example to show that this is false. That is, give an example of a function f and a point x where f is increasing but $f'(x)$ is not positive.

2. Some people think that if a function is continuous at a point, then it is differentiable at that point.

a. Is this true or false? Explain. **b.** Show that this is false.

c. Give an example to show that this is false. That is, give an example of a function f and a point x so that f is continuous at x but f is not differentiable at x.

Bridging the Gap: Both Ends Are Important

3. Some people say that $\lim_{n\to\infty} a_n = 0$ means that the numbers a_n get closer and closer to zero as *n* gets larger and larger. Find an example of a sequence $a_1, a_2, a_3, \ldots$ that satisfies this condition but for which the limit is not zero.

4. Some people say that $\lim_{n\to\infty} a_n = 0$ means that the numbers a_n get as close as you like to zero but never equal zero as *n* gets larger and larger. Find an example of a sequence $a_1, a_2, a_3, \ldots$ where the limit is zero but where some of the a_n also equal zero.

5. Some people say that $\lim_{n\to\infty} a_n = 0$ means that given any specified degree of closeness to zero, you can always find values of a_n that are at least that close to zero. Find an example of a sequence $a_1, a_2, a_3, \ldots$ that satisfies this condition but where the limit is not zero.

My Philosophy in a Nutshell

- Teach logical reasoning, not just logic as a subject. Build bridges between ordinary and mathematical language
- Be conscious of the tension between covering topics and developing students' understanding
- Be aware that many students today are not very good at algebra
- Don't rush to present topics from an advanced perspective
- Interaction is essential
- Always be respectful; never act surprised
- For most students, learning to prove is a long-term process – follow-up in subsequent courses is essential

Some References

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2. The Role of Logic in Teaching Proof. *The American Mathematical Monthly* (110)10, 2003 (http://condor.depaul.edu/sepp/monthly886-899.pdf)

3. Variables in Mathematics Education. In *Tools for Teaching Logic*. Blackburn, P. et al., Eds. Springer Publishing, 2011 (Reprinted in *The Best Writing on Mathematics 2012*, M. Pitici, Ed., Princeton Univ. Press, 2012 (http://condor.depaul.edu/sepp/VariablesInMathEd.pdf)

Additional papers are available at http://condor.depaul.edu/sepp.