Counting Melodies

A Musical Introduction to Recursion

Kurt Ludwick
Salisbury University
Salisbury, MD 21801

keludwick@salisbury.edu
http://faculty.salisbury.edu/~keludwick

Counting Melodies

A Musical Introduction to Recursion

- Math 105: Liberal Arts Mathematics,
  Music & Mathematics

- Target audience – liberal arts students who need one math course to complete general education requirements

- Generally math-averse/anxious/hostile, wide range of levels of mathematical preparation/retention.

Counting Melodies
A Musical Introduction to Recursion

SU DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
SYLLABUS *(Tentative)*
MATH 105: *Liberal Arts Mathematics: Music and Mathematics*

OBJECTIVES: To introduce students to some of the many connections between mathematics and music, and to explore mathematical questions that follow naturally from standard musical considerations such as intonation, melody, rhythm, and variations on a theme.

INTENDED FOR: Liberal Arts Majors, particularly those with an interest in music.

PREREQUISITES: High School Algebra II or a college algebra course. *(Some experience with music – in particular, the ability to read music from a staff – is preferred, but it is not a strict requirement for this course.)*

TEXT: "The Math Behind the Music," by Leon Harkleroad; Cambridge University Press, First Edition, 2006.

TECHNOLOGY: A basic scientific calculator (not necessarily a graphing calculator) that handles exponents and logarithms. *(This type of calculator usually costs between $10 and $20. DO NOT invest a large amount of money in a calculator if it is to be used only for this class.)*

Counting Melodies
A Musical Introduction to Recursion

Topics	Weeks
Pitch *Frequency; octaves and other intervals; overtones*	**1**
Intonation *The twelve-tone scale; Pythagorean tuning; just intonation; equal-tempering; alternate divisions of the octave*	**3**
Variations *Transpositions; retrogrades; inversions; musical operations; groups*	**3**
Counting *Combinations and permutations; the multiplication principle; counting melodies and chords; time signature and rhythm*	**3**
Bells and Groups *Change-ringing; permutations; patterns, subgroups and cosets*	**2**
Additional Topics (as time permits) *Patterns in music; randomized music; music and geometry; electronic music*	**1**
Tests, Review & Other Activities	**1**

Counting Melodies

A Musical Introduction to Recursion

- How many different n-beat rhythms can be written using quarter notes (1 beat), half notes (2 beats), dotted half notes (3 beats), or whole notes (4 beats)?

Counting Melodies
A Musical Introduction to Recursion

- How many different n-beat rhythms can be written using quarter notes (1 beat), half notes (2 beats), dotted half notes (3 beats), or whole notes (4 beats)

n

1

(one 1-beat rhythm)

2

(two 2-beat rhythms)

3

(four 3-beat rhythms)

Counting Melodies
A Musical Introduction to Recursion

- How many different n-beat rhythms can be written using quarter notes (1 beat), half notes (2 beats), dotted half notes (3 beats), or whole notes (4 beats)?

n

1

(one 1-beat rhythm)

2

(two 2-beat rhythms)

3

(four 3-beat rhythms)

4

…eight 4-beat rhythms.

…so, how many 5-beat rhythms do we expect to see?

Counting Melodies

A Musical Introduction to Recursion

- How many different n-beat rhythms can be written using quarter notes (1 beat), half notes (2 beats), dotted half notes (3 beats), or whole notes (4 beats)?

Let x_n denote the number of n-beat rhythms that can be written using quarter, half, dotted half, or whole notes…

n	x_n
1	1
2	2
3	4
4	8

Counting Melodies
A Musical Introduction to Recursion

- How many different n-beat rhythms can be written using quarter notes (1 beat), half notes (2 beats), dotted half notes (3 beats), or whole notes (4 beats)?

Let x_n denote the number of n-beat rhythms that can be written using quarter, half, dotted half, or whole notes…

n	x_n
1	1
2	2
3	4
4	8
5	15
6	29
7	56

…?

Counting Melodies

A Musical Introduction to Recursion

- Similar question:
  How many different n-beat rhythms can be written using only quarter notes and half notes?

n

1 

$x_1 = 1$

2 

$x_2 = 2$

3 

$x_3 = 3$

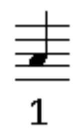
So… what's the pattern? $x_n = n$?

Counting Melodies
A Musical Introduction to Recursion

- Similar question:
  How many different n-beat rhythms can be written using only quarter notes and half notes?

n

1



$x_1 = 1$

2

$x_2 = 2$

3

$x_3 = 3$

4

$x_4 = 5$

So… that hypothesis didn't work. What's happening here?

Counting Melodies

A Musical Introduction to Recursion

n	1	2	3	4	5	6	7	8
x_n	1	2	3	5				

To figure out the pattern, we'll calculate x_5 by methodically finding each allowable five-beat rhythm.

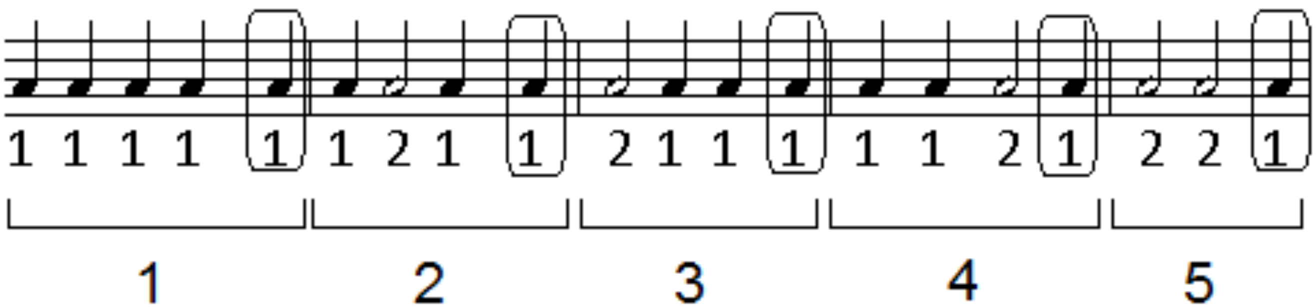
First, observe that the last note of each rhythm must be either a quarter note (1 beat) or a half note (2 beats)…

Counting Melodies

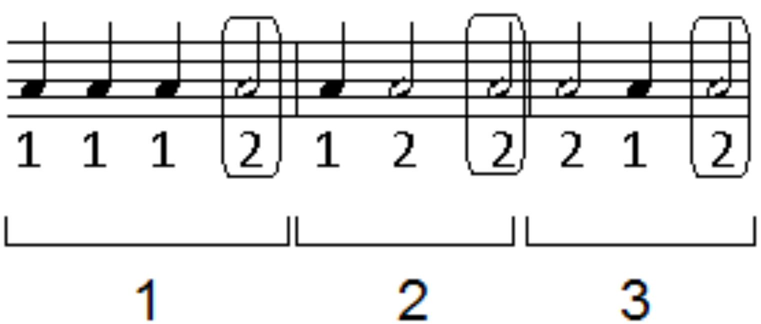
A Musical Introduction to Recursion

n	1	2	3	4	5	6	7	8
x_n	1	2	3	5				

Rhythms ending with a quarter note…



Rhythms ending with a half note…

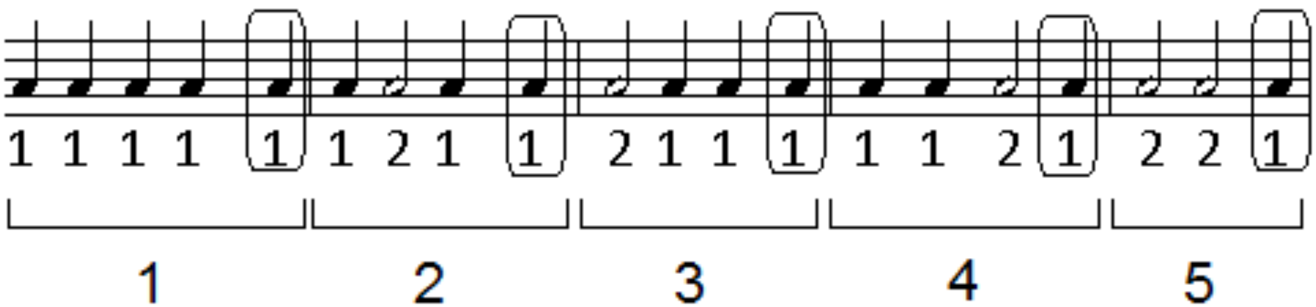


Counting Melodies

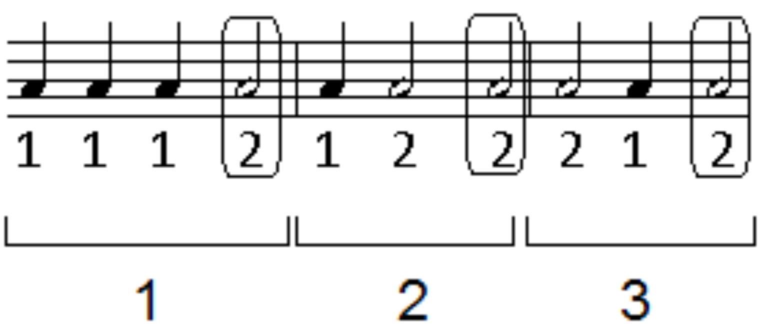
A Musical Introduction to Recursion

n	1	2	3	4	5	6	7	8
x_n	1	2	3	5	8			

Rhythms ending with a quarter note…



Rhythms ending with a half note…



So, $x_5 = 5 + 3 = 8.$

$\uparrow$ $\uparrow$

x_4 x_3

Counting Melodies

A Musical Introduction to Recursion

Summary: The number of five-beat rhythms is equal to the number of four-beat rhythms plus the number of three-beat rhythms. In other "words," $x_5 = x_4 + x_3$. We can see why this happens...

Four-beat rhythms...
($x_4 = 5$)

...plus a quarter note...

...give us all of the five-beat rhythms that end in a quarter note.

Similarly,
three-beat rhythms...
($x_3 = 3$)
...plus a half note...

...give us all of the five-beat rhythms that end in a half note.

Counting Melodies

A Musical Introduction to Recursion

It's not hard to see how this generalizes…

To find all of the n-beat rhythms (where $n > 2$),

- Add a quarter note to each $n - 1$ beat rhythm, and
- Add a half note to each $n - 2$ beat rhythm.

Since only half notes and quarter notes are allowed, this process will give us each n-beat rhythm, without omitting or repeating any.

Therefore, for all $n > 2$, $x_n = x_{n-1} + x_{n-2}$, which gives us the Fibonacci sequence!

n	1	2	3	4	5	6	7	8	9	10	11	12 …
x_n	1	2	3	5	8	13	21	34	55	89	144	233 …

Counting Melodies

A Musical Introduction to Recursion

Generalizations of this idea…

- Back to the first example (allowing quarter, half, dotted-half, or whole notes)

Let x_n denote the number of n-beat rhythms that can be written using quarter, half, dotted-half, or whole notes…

n	x_n
1	1
2	2
3	4
4	8
5	15
6	29
7	56

By a similar line of reasoning, we end up with a four-term recursion:

$$x_n = x_{n-1} + x_{n-2} + x_{n-3} + x_{n-4}$$

Counting Melodies

A Musical Introduction to Recursion

Generalizations of this idea…

- Allow only quarter notes (1 beat) and dotted-half notes (3 beats)…

one 1-beat rhythm

one 2-beat rhythms

two 3-beat rhythms

three 4-bear rhythms

We end up with the three-term recursion $x_n = x_{n-1} + x_{n-3}$

n	1	2	3	4	5	6	7	8	9	10	11	12	…
x_n	1	1	2	3	4	6	9	13	19	28	41	60	

Counting Melodies
A Musical Introduction to Recursion

Generalizations of this idea…

- How many n-beat melodies can we write using only quarter notes and half notes, where each note may take on one of two pitches – say, a "G" or an "A?"

  Let x_n denote the number of such n-beat melodies…

$x_1 = 2$

$x_2 = 6$

Counting Melodies

A Musical Introduction to Recursion

Generalizations of this idea…

- How many n-beat melodies can we write using only quarter notes and half notes, where each note may take on one of two pitches?

$x_1 = 2$

$x_2 = 6$

Each 3-beat melody must correspond to one of the following:

2-beat melody, plus 

$x_2 = 6$ melodies

1-beat melody, plus 

$x_1 = 2$ melodies

2-beat melody, plus 

$x_2 = 6$ melodies

1-beat melody, plus 

$x_1 = 2$ melodies

Therefore: $x_3 = 2x_2 + 2x_1 = 2 \cdot 6 + 2 \cdot 2 = 16$

Counting Melodies

A Musical Introduction to Recursion

Generalizations of this idea…

- How many n-beat melodies can we write using only quarter notes and half notes, where each note make take on one of two pitches?

General Solution: $x_1 = 2, x_2 = 6$, and for $n > 2$ we use the two-term recursion $x_n = 2x_{n-1} + 2x_{n-2}$

n	1	2	3	4	5	6 …
x_n	2	6	16	44	120	328 …

Counting Melodies
A Musical Introduction to Recursion

- A variety of similar examples can be used to demonstrate different types of recursive sequences. As we've seen, this is not limited to two-term recursions.

- The quarter/half note rhythm example gives one more way (among many others) to demonstrate how the Fibonacci sequence (and similar sequences) arise naturally in the arts

- Students tend to eventually understand the basic idea of recursion in general, and how to look for different types of patterns in sequences. (We cover this along with arithmetic and geometric sequences.)

- Students tend to struggle with the x_n notation -- verbal descriptions are preferred (and are usually – though not always – adequate)

Counting Melodies

A Musical Introduction to Recursion

THANK YOU!

To the MAA for supporting this session, and to Dr. Sarah Mabrouk for organizing it and for including this talk…

To Salisbury University, and in particular my colleagues in the Department of Mathematics & Computer Science, for freedom and flexibility in developing and teaching the Math & Music course…

To you, the audience, for your presence, attention, and for any questions you may have!

Counting Melodies

A Musical Introduction to Recursion

THAND YOU!

Have a nice day.

Questions…?

Kurt Ludwick
Salisbury University
Salisbury, MD 21801

keludwick@salisbury.edu
http://faculty.salisbury.edu/~keludwick